

Fermion Determinant and the Sphaleron Bound

Guy D. Moore¹

*Princeton University
Joseph Henry Laboratories, PO Box 708
Princeton, NJ 08544, USA*

Abstract

We investigate analytically the fermionic fluctuation determinant at finite temperatures in the minimal standard model, including all operators up to dimension 6 and all contributions to the effective potential to all orders in the high T expansion, to 1 loop. We apply the results to find corrections to the Sphaleron erasure rate in the broken phase. We conclude that the analytic treatment of fermions is very reliable, and that there is a great deal of baryon erasure after the phase transition for any physical Higgs mass.

1 Introduction

It has been known since the pioneering work of Kirzhnits and Linde [1] and Dolan and Jackiw [2] that, at high temperature, the thermal populations of massive particles exert a symmetry restoring force on the Higgs condensate, so that at sufficiently high temperature ($T \sim m_H/g$) the Higgs field loses its condensate and electroweak symmetry is restored. Under these circumstances baryon number is readily violated [3, 4]. As temperature falls there is a symmetry breaking phase transition, which is first order for small m_H . Presumably the universe underwent such a phase transition shortly after the Big Bang, and it is believed likely that the baryon asymmetry of the universe was generated at this time.

Equilibrium thermodynamic information important to the physics of this epoch, such as the effective potential and the free energy of certain saddlepoint solutions, can be computed in the Matsubara (imaginary time) formalism. The particular calculation we will be interested in is the Sphaleron rate in the broken phase shortly after the phase transition. This was first investigated at tree level in [5], and one loop corrections have been computed within a high temperature approximation in [6, 7]. This approximation assumes that the contributions of nonzero Matsubara frequencies can be absorbed into shifts in the parameters of the zero Matsubara frequency modes, which then constitute a “dimensionally reduced” three dimensional theory[8]. The same idea forms the basis for an extensive investigation of the effective potential and the strength of the phase transition being carried out by Farakos, Kajantie, Laine, Rummukainen, and Shaposhnikov (FKLRS) [9, 10].

Why is it necessary to include 1 loop corrections? Normally, one loop corrections modify tree level results by only a few percent, but there are exceptions. The finite temperature, 1 loop correction to the Higgs mass is significant because, although the correction is parametrically order g^2 , the mass is a dimensionful parameter, and the importance of the thermal

¹e-mail: guymoore@puhep1.princeton.edu

effect is enhanced by T^2/m^2 , which can be large. The phase transition temperature is the point where the thermal, 1 loop effect and the tree level effect are almost equal. Finite temperature calculations also have potential infrared divergences from loops containing zero Matsubara frequency bosons, and it is not clear that perturbation theory always works; the object of the dimensional reduction program is to separate this problem from those effects which can be treated perturbatively more reliably. Finally, loop corrections can also be important when a coupling is unnaturally small; for instance, the Higgs self coupling λ receives a 1-loop correction which goes as g_t^4 (g_t is the Yukawa coupling of the top quark). In this paper we will always take $g_t = 1$, corresponding to $m_t \simeq 174\text{GeV}$). This correction does not depend on λ , and if λ happens to be small then the correction can be very important. This is the case in the standard model with a light Higgs boson and a heavy top quark. It is quite necessary, then, to consistently use 1 loop relationships when describing anything directly related to λ , such as the physical Higgs mass and the (finite and zero temperature) effective potential. In this case, however, two loop effects are still expected to make a small perturbative correction to the one loop effects, so the one loop calculation should be reliable.

With this in mind, Diakonov, Polyakov, Sieber, Schaldach, and Goeke (DPSSG) have made a direct numerical evaluation of the fermionic fluctuation determinant and have concluded on its basis that, when the top quark is heavy, fermion fluctuations make the Sphaleron rate very different from the high temperature estimate [11]. This brings the dimensional reduction program into question, and bears further investigation. Here we give a one loop, perturbative treatment of the fermions in the presence of zero Matsubara frequency background fields. The free energy of the fermions can naturally be expressed in terms of operators made up of zero Matsubara frequency bosonic fields and their derivatives. In section II we compute these, including induced masses (dimension 2), corrections to couplings and wave function renormalizations (dimension 4), and nonrenormalizable dimension 6 operators. The expansion in operator dimension appears to be very well behaved when the background bosonic fields are slowly varying. In section III we apply these results to a computation of the Sphaleron energy. In section IV we use this calculation, along with 1-loop relationships between the couplings and the physical Higgs mass and a calculation of the bubble nucleation rate which determines the temperature at which the phase transition occurs, to investigate the erasure of baryons after the phase transition is complete. In the last section we draw conclusions. Some technical details are relegated to an appendix.

2 Integration over fermions

We will work in SU(2) Higgs theory coupled to all the fermions of the standard model. Neglecting hypercharge significantly simplifies the picture without profoundly changing it, because the Weinberg angle is small and because the fields which make up the Sphaleron are almost exclusively weak isospin, and not hypercharge, fields. (It is known that, at tree level, including hypercharge with the physical Weinberg angle only changes the Sphaleron energy by about 1% [5]). The Lagrangian, suppressing summations on generations and

colors, is

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_b + \mathcal{L}_f \\ \mathcal{L}_b &= \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + (D_\mu \Phi)^\dagger D_\mu \Phi - m_0^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \\ \mathcal{L}_f &= \bar{\psi}_L \gamma_\mu D_\mu \psi_L + \bar{\psi}_R \gamma_\mu \partial_\mu \psi_R + g_t (\bar{\psi}_L \Phi \psi_R + \bar{\psi}_R \Phi^\dagger \psi_L).\end{aligned}$$

(We use a Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$ and Euclidean γ matrices which satisfy the algebra $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$.) The fermion kinetic terms apply for every doublet and singlet, but the mass term only applies for the top quark. Actually the mass term as written is for a bottom, not top, type quark. This is for notational simplicity only; to make the top quark massive one should systematically replace Φ with $i\tau_2 \Phi^*$ in all that follows; our conclusions are completely unchanged.

In the 1 loop approximation we ignore the fermions' coupling to nonzero Matsubara frequency excitations of the bosonic fields. The path integral over the fermions is gaussian; its contribution to the partition function is $\text{Det}H$, and the corresponding contribution to the effective action is $-\text{Tr} \ln H$, where H , written as a matrix acting on $[\psi_L^\alpha \ \psi_R^\alpha]^T$, is

$$H = \begin{bmatrix} \gamma_\mu D_{\beta\alpha}^\mu & g_t \Phi_\beta \\ g_t \Phi_\alpha^\dagger & \gamma_\mu \partial^\mu \end{bmatrix} = \begin{bmatrix} \gamma_\mu \partial^\mu \delta_{\beta\alpha} & 0 \\ 0 & \gamma_\mu \partial^\mu \end{bmatrix} + \begin{bmatrix} \frac{ig_w}{2} \gamma_\mu A_a^\mu \tau_{\beta\alpha}^a & g_t \Phi_\beta \\ g_t \Phi_\alpha^\dagger & 0 \end{bmatrix} \equiv H_0 + H_I, \quad (1)$$

where α and β are $SU(2)$ indices.

Our analysis will be based on an expansion of $-\text{Tr} \ln H$ in H_I . To illustrate the idea of expanding the log, consider a simplified example in which the gauge field is everywhere zero. In this case we have a Dirac fermion with a (spatially varying) mass, $H = \gamma^\mu \partial_\mu - m$, where $m^2 = g_t^2 \Phi^\dagger \Phi$. We assume that m^2 is smaller than the lowest eigenvalue of $-\partial^2$, in which case it is legitimate to expand the log. This will generally be the case, as the lowest eigenvalue of $-\partial^2$ is set by the square of the lowest possible Matsubara frequency, which for a fermion is $(\pi T)^2$. The log becomes

$$-\text{Tr} \ln(\gamma^\mu \partial_\mu - m) = -\text{Tr} \ln \gamma^\mu \partial_\mu - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr} \left(\frac{m}{\gamma_\mu \partial_\mu} \right)^n \quad (2)$$

where as usual $1/(\gamma^\mu \partial_\mu)$ is defined as $\gamma^\mu \partial_\mu / \partial^2$. We get Feynman diagrams by the usual trick of inserting complete sets of states and Fourier transforming. The first term in the sum is a divergent vacuum energy and should be removed. All terms with odd powers of m vanish when we take the trace on Dirac indices. When the mass is position independent, the resulting terms are

$$\int_0^{T^{-1}} dx_0 \int d^3x \sum_{n=1}^{\infty} (-)^n \frac{4m^{2n}}{2n} T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + k_0^2)^n}. \quad (3)$$

We have used the shorthand \sum_{k_0} to mean a sum in which k_0 takes on odd integer multiples of πT ; we will use this shorthand throughout. Also, k^2 means \vec{k}^2 and k means $\sqrt{\vec{k}^2}$. The $(-)^n$ arises because when we Fourier transform $\partial \rightarrow ik$. The 4 is from the Dirac trace, and

$2n$ is the symmetry factor of the diagram; we saw how it arises in the expansion of the log above. To get the free energy density we drop the space integral.

The first two integrals in the series are ultraviolet divergent and must be performed with some care. We have done so in the $\overline{\text{MS}}$ scheme by first performing the sum on Matsubara modes and conducting the spatial integrals in $3-2\epsilon$ dimensions. The higher order terms are convergent and may be performed directly by doing the integral over d^3k first. Performing the integrals, the free energy per unit volume is

$$m^2 \frac{T^2}{12} + m^4 \frac{2\gamma_E - 2\ln\pi + \ln(\mu^2/T^2)}{16\pi^2} + \sum_{n=3}^{\infty} (-)^n \frac{4m^{2n}}{2n} \frac{(1 - 2^{3-2n})\zeta(2n-3)\Gamma(n-\frac{3}{2})}{4\pi^{2n-2}T^{2n-4}\Gamma(\frac{1}{2})\Gamma(n)} \quad (4)$$

with μ the usual $\overline{\text{MS}}$ renormalization point. This is the 1 loop fermionic contribution to the finite temperature effective potential, to all orders in the high temperature expansion². Note that, as expected, the sum is a Taylor series in m^2 with radius of convergence $(\pi T)^2$. By m_t we mean the thermal top quark mass given by $m_t^2(T) = g_t^2 \Phi^\dagger \Phi(T)$, not the vacuum top quark mass $m_t^2(0) = g_t^2 \Phi^\dagger \Phi(T=0)$. Near the phase transition temperature the difference is quite substantial. In the simplest approximation $\Phi^\dagger \Phi(T) = \Phi^\dagger \Phi(T=0)(1 - T^2/T_c^2)$, but near T_c it is necessary to treat the influence of infrared bosons more carefully. As we will see in Sec. IV, at the temperature of interest the Higgs VEV $\nu \equiv \sqrt{\Phi^\dagger \Phi/2} \leq T$ and for $g_t = 1$ we find $(m/\pi T)^2 \leq 1/(2\pi^2)$, so the convergence of the series is excellent. Of course, well below the phase transition temperature it is no longer true that $m_t \ll \pi T$ and our approximation scheme will fail. However, at these temperatures the Sphaleron energy is so large that the baryon erasure rate is utterly negligible, so our approximation scheme should apply in the interesting range of temperatures.

Another way of understanding the convergence of the power series is to think of the different Matsubara frequency contributions of the top quarks as distinct, very massive species of a three dimensional theory, with masses given by the Matsubara frequencies $m_M^2 = ((2n+1)\pi T)^2$; we are then expanding in the ratio of $g_t^2 \Phi^\dagger \Phi$ (and eventually in products of other infrared bosonic fields like A_i , and their derivatives) to m_M^2 . Such an expansion has been considered in a theory with only fermions and a scalar in [12, 13], where it is also concluded that such an expansion is very accurate. An expansion like the one used here is not justified for bosons because symmetric boundary conditions in time make the lowest eigenvalue of the operator ∂^2 is zero; or equivalently the lowest Matsubara frequency is zero, so we cannot expand in the ratio of a field value to this Matsubara frequency. (However we could expand all the other Matsubara frequencies in the way described here, see [9].)

For simplicity of notation, in the remainder of the paper we write

$$T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + k_0^2)^2} = \frac{2\gamma_E - 2\ln\pi + \ln(\mu^2/T^2)}{16\pi^2} \equiv D4 \quad (5)$$

and

$$T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + k_0^2)^N} = \frac{(1 - 2^{3-2N})\zeta(2N-3)\Gamma(N-\frac{3}{2})}{4\pi^{2N-2}T^{2N-4}\Gamma(\frac{1}{2})\Gamma(N)} \equiv D2N. \quad (6)$$

²An integral expression for the fermionic contribution was first found by Dolan and Jackiw [2], who also found the first two terms in the expansion given here. What we have done is found the complete Taylor series for the known integral expression.

in particular,

$$D6 = \frac{7\zeta(3)}{128\pi^4 T^2}. \quad (7)$$

When the mass is spatially varying the first term in the sum in Eq.(3) becomes

$$\int dx_0 \frac{-1}{2} \int \frac{d^3 p}{(2\pi)^3} m(p) m(-p) T \sum_{k_0} \int \frac{d^3 k}{(2\pi)^3} \frac{\text{tr}(\gamma^\mu \gamma^\nu) k_\mu (p+k)_\nu}{(k^2 + k_0^2)((p+k)^2 + k_0^2)}. \quad (8)$$

If we assume that $p^2 < k^2 + k_0^2$, which means that the mass varies on a scale large compared to $1/\pi T$, then we may expand the denominator in a geometric series and extract a power series in p^2 . We can then Fourier transform to position space to express the result as a derivative expansion for m . The resulting free energy is

$$\int d^3 x \left(\frac{T^2}{12} m^2 + D4 (\vec{\nabla} m)^2 - \frac{D6}{3} (\nabla^2 m)^2 + \frac{D8}{10} (\vec{\nabla} \nabla^2 m)^2 \dots \right). \quad (9)$$

In realistic cases the Fourier transform of m lies primarily at small values of p but has a rapidly decaying exponential tail which goes above $p = \pi T$. In this case the series will be asymptotic and its reliability will depend on how rapidly the exponential tail falls off. For the Sphaleron at $T \simeq T_c$ we expect the exponential tail of the Fourier transform of the gauge and Higgs field configurations to fall roughly as $\exp(-p/m_W(T))$; at $T \simeq T_c$, $m_W(T) = g_w \nu(T)/2$ is $\ll \pi T$ and the convergence of the (asymptotic) derivative expansion should be excellent. Of course we will check this by explicit calculation.

For the more general H the lowest nonvanishing term will be

$$\begin{aligned} \frac{1}{2} \text{Tr} \frac{H_I}{H_0^2} H^0 \frac{H_I}{H_0^2} H_0 &\Rightarrow \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} T \sum_{k_0} \int \frac{d^3 k}{(2\pi)^3} (\Phi_\alpha^\dagger(p) \Phi_\alpha(-p)) \frac{i k_\mu i (k+p)_\nu \text{tr}(\gamma^\mu \gamma^\nu)}{(k^2 + k_0^2)((k+p)^2 + k_0^2)} \\ &+ \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} T \sum_{k_0} \int \frac{k^3}{(2\pi)^3} \left(\frac{g_w^2 \delta_{ab}}{2} i A_\mu^a(p) i A_\nu^b(-p) \right) \frac{i k_\alpha i (p+k)_\beta \text{tr}(\frac{1-\gamma^5}{2} \gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu)}{(k^2 + k_0^2)((k+p)^2 + k_0^2)} \end{aligned} \quad (10)$$

which corresponds to the Feynman diagrams illustrated in Fig. 1. Higher order contributions can be gotten similarly by going to higher powers in H_I .

The contribution of fermionic fluctuations to dimension two and four operators in $SU(2) \times U(1)$ Higgs theory in $\overline{\text{MS}}$ with realistic couplings has recently been worked out by FKLRS [10]. We independently performed the calculation in $SU(2)$ Higgs theory; our results concur. We have also extended the calculation to find all nonrenormalizable operators induced by fermions at dimension 6. Our results follow.

The dimension two operators induced by fermions change the masses of the Higgs and A_0 fields. The induced terms are

$$\frac{N_c g_t^2 T^2}{12} \Phi^\dagger \Phi + \frac{N_d g_w^2 T^2}{24} A_0^2. \quad (11)$$

The first term is largely responsible for the restoration of symmetry at high temperature. The second is the familiar Debeye screening mass. (Here and throughout $N_c = 3$ is the number of colors and $N_d = 12$ is the number of left handed fermion doublets.)

At dimension four, fermions introduce the following corrections to couplings and wave functions:

$$N_c g_t^4 D4 (\Phi^\dagger \Phi)^2 + N_c g_t^2 D4 (D_i \Phi)^\dagger D_i \Phi + \frac{N_c g_w^2 g_t^2}{4} (D4 - \frac{1}{8\pi^2}) A_0^2 \Phi^\dagger \Phi \\ + \frac{N_d g_w^2 D4}{12} F_{ij}^a F_{ij}^a + \frac{N_d g_w^2}{6} (D4 - \frac{1}{16\pi^2}) (D_i A_0)^a (D_i A_0)^a - \frac{N_d g_w^4}{192\pi^2} (A_0^2)^2. \quad (12)$$

The coefficients of $D4$ give the fermionic contributions to the one loop beta functions and anomalous dimensions for the bosonic fields in $SU(2)$ Higgs theory, and agree with well known results. If the dimensional reduction scheme is justified then these terms should give an accurate approximation of the fluctuation determinant. To check this it is necessary to go one order higher and see if the contributions of dimension 6 operators are small.

At dimension six, fermions contribute the following terms to the effective potential,

$$- \frac{2N_c g_t^6 D6}{3} (\Phi^\dagger \Phi)^3 + \frac{N_c g_t^2 g_w^4 D6}{16} (A_0^2)^2 \Phi^\dagger \Phi, \quad (13)$$

the following Higgs derivative terms,

$$- \frac{4N_c g_t^4 D6}{3} (\Phi^\dagger \Phi) (D_i \Phi)^\dagger D_i \Phi - \frac{2N_c g_t^4 D6}{3} (\Phi^\dagger D_i \Phi) (D_i \Phi)^\dagger \Phi \\ - \frac{N_c g_t^4 D6}{3} \partial_i (\Phi^\dagger \Phi) \partial_i (\Phi^\dagger \Phi) - \frac{N_c g_t^2 D6}{3} (D^2 \Phi)^\dagger D^2 \Phi, \quad (14)$$

the following mixed derivative terms,

$$\frac{N_c g_w^2 g_t^2 D6}{6} A_0^2 (D_i \Phi)^\dagger D_i \Phi - \frac{N_c g_w^2 g_t^2 D6}{12} (D_i A_0)^a (D_i A_0)^a \Phi^\dagger \Phi, \quad (15)$$

the following A_0 derivative terms,

$$\frac{N_d g_w^4 D6}{24} \partial_i (A_0^2) \partial_i (A_0^2) - \frac{N_d g_w^2 D6}{30} (D^2 A_0)^a (D^2 A_0)^a, \quad (16)$$

the following mixed terms,

$$- \frac{N_c g_t^2 g_w^2 D6}{8} F_{ij}^a F_{ij}^a \Phi^\dagger \Phi - \frac{N_d g_w^4 D6}{24} A_0^2 F_{ij}^a F_{ij}^a + \frac{N_d g_w^4 D6}{8} A_0^a F_{ij}^a A_0^b F_{ij}^b, \quad (17)$$

and the following gauge field terms,

$$- \frac{N_d g_w^3 D6}{180} f_{abc} F_{ij}^a F_{jk}^b F_{ki}^c - \frac{N_d g_w^2 D6}{15} (D_i F_{ij})^a (D_k F_{kj})^a. \quad (18)$$

Fermionic contributions to other independent gauge invariant dimension 6 operators (such as $(A_0^2)^3$) vanish.

Exactly the same terms arise whether the massive quark is top type or bottom type. If both types are massive (in one generation) then each occurrence of g_t^n becomes $g_t^n + g_b^n$ in the above, and exactly one mixed term appears at dimension 6,

$$- \frac{4N_c g_t^2 g_b^2 D6}{3} \Phi^\dagger D_j i\tau_2 \Phi^* (D_j i\tau_2 \Phi^*)^\dagger \Phi. \quad (19)$$

Fermions also induce pure QCD operators and operators containing both weak and strong fields; these are unimportant here and are listed in Appendix B.

We should comment that, except when they coincidentally vanish, the fermionic contributions to dimension six operators tend to be much larger than the contributions from the nonzero Matsubara frequencies of boson fields, some of which have been worked out in [14]. This is partly because the top quark is heavier than any of the bosonic degrees of freedom and partly because its lowest nonzero Matsubara frequency is πT , rather than $2\pi T$. Because of this the bosonic equivalent of the coefficient $D6$ is smaller by a factor of 7. Hence, we anticipate that if the expansion in high dimension operators is well behaved for fermions then it will also be well behaved for bosons.

We have also computed the fermionic contributions to masses, couplings, and wave function renormalizations in a slight modification of the proper time technique of DPSSG. The procedure is outlined in Appendix A. The results turn out to be identical to those in $\overline{\text{MS}}$ except that $D4$ is modified to

$$D4_{\text{proper time}} = \frac{\gamma_E - 2 \ln \pi + \ln(\mu^2/T^2)}{16\pi^2}, \quad (20)$$

where the definition of the scale μ is explained in Appendix A. This expression relates the $\overline{\text{MS}}$ and proper time renormalization points. The appendix also presents the calculation of the vacuum effective potential and the physical Higgs mass in the proper time scheme.

3 The Sphaleron

We want to apply these results to find the free energy of a nontrivial field configuration which solves the classical equations of motion. Klinkhamer and Manton have shown [5] that the classical equations of motion of the $SU(2)$ Higgs system can be solved by an *Ansatz* of form

$$\Phi = \frac{\nu}{\sqrt{2}} \frac{h(r)}{r} \begin{bmatrix} x + iy \\ z \end{bmatrix}, \quad A_i^j = \frac{2f(r)}{r^2} r_k \epsilon_{ijk}. \quad (21)$$

The lower index on A is the Lorentz index and the upper index is the group index, and h and f are functions of r alone which are to be determined by minimizing the configuration's energy, subject to the boundary conditions $f(0) = h(0) = 0$ and $f(\infty) = h(\infty) = 1$. (Here and throughout ν is the Higgs VEV, $\nu^2(T) = 2\Phi^\dagger\Phi(T)$.) This solution is called the Sphaleron, and when $E_{Sph} \gg T$, the formalism of Langer tells us that most baryon number violating events should occur because of phase space paths which pass close to this configuration. Arnold and McLerran have applied this idea to estimate the rate of baryon number violation in the broken electroweak phase to be [4]

$$\frac{dN_B}{N_B dt} = -13N_F T \left(\frac{\alpha_w}{4\pi} \right)^4 \frac{\omega_-}{2m_W} \left(\frac{4\pi\nu}{g_w T} \right)^7 \mathcal{N}_{\text{tr}} \mathcal{N} \mathcal{V}_{\text{rot}} \kappa \exp(-E_{Sph}/T) \quad (22)$$

with $N_F = 3$ the number of generations. The prefactors arise from the zero and unstable modes of the Sphaleron, and are evaluated in [15, 6]. At 1 loop κ is a product of fluctuation

determinants around the configuration,

$$\kappa = \frac{\text{Det}H}{\text{Det}H_0} \left(\frac{\text{Det}K_0}{\text{Det}K'} \right)^{\frac{1}{2}}. \quad (23)$$

H is Eq. (1) in the Sphaleron background and H_0 is Eq. (1) in the naive vacuum. K_0 is the bosonic fluctuation determinant in vacuum, and K' is the bosonic determinant in the Sphaleron background, but with the zero and unstable modes removed.

We should comment that the division between $\ln \kappa$ and $-E_{sph}/T$ is somewhat arbitrary. In particular it is renormalization point dependent. We see this explicitly from our calculation of the contributions from fermions to dimension 4 operators (Eq. (12)), which are part of $-\ln \kappa$, but which have with coefficients $\propto D4$ which explicitly depend on μ . The calculation of E_{Sph} correspondingly depends on coupling constants g_w, λ which depend on μ .

To compute $E_{Sph} - T \ln \kappa$ we need to evaluate the tree level Lagrangian terms $\Phi^\dagger \Phi$, $(\Phi^\dagger \Phi)^2$, and $F_{ij}^a F_{ij}^a$, and the operators found in Section II, in the Sphaleron background, Eq. (21). First note that $A_0 = 0$, so all terms including A_0 vanish. Effective potential terms give

$$\Phi^\dagger \Phi = \frac{\nu^2}{2} h^2, \quad (24)$$

$$(\Phi^\dagger \Phi)^2 = \frac{\nu^4}{4} h^4, \quad (25)$$

$$(\Phi^\dagger \Phi)^3 = \frac{\nu^6}{8} h^6 \dots . \quad (26)$$

We will also need the term $(\Phi^\dagger \Phi)^{3/2} = \nu^3 h^3 / (2\sqrt{2})$ which arises from bosonic fluctuations, and we should add a $\Phi^\dagger \Phi$ independent constant to make the global minimum of $V(\nu)$ zero, to subtract out the energy density in the absence of a Sphaleron.

The dimension 4 derivative terms are [5]

$$F_{ij}^a F_{ij}^a = 16 \frac{f'^2}{r^2} + 32 \frac{f^2(1-f)^2}{r^4}, \quad (27)$$

$$(D_i \Phi)^\dagger D_i \Phi = \frac{\nu^2}{2} \left(2 \frac{h^2(1-f)^2}{r^2} + h'^2 \right). \quad (28)$$

We calculate that the dimension 6 derivative terms are

$$f_{abc} F_{ij}^a F_{jk}^b F_{ki}^c = 96 \frac{f'^2 f(1-f)}{r^4}, \quad (29)$$

$$(D_i F_{ij})^a (D_k F_{kj})^a = 8 \frac{f''^2}{r^2} - 32 \frac{f'' f(1-f)}{r^4} + 32 \frac{f^2(1-f)^2}{r^6}, \quad (30)$$

$$\Phi^\dagger \Phi F_{ij}^a F_{ij}^a = \frac{\nu^2 h^2}{2} \left(16 \frac{f'^2}{r^2} + 32 \frac{f^2(1-f)^2}{r^4} \right), \quad (31)$$

$$(D^2 \Phi)^\dagger D^2 \Phi = \frac{\nu^2}{2} \left(h'' + 2 \frac{h'}{r} - 2 \frac{h(1-f)^2}{r^2} \right)^2, \quad (32)$$

$$\Phi^\dagger \Phi (D_i \Phi)^\dagger D_i \Phi = \frac{\nu^4}{4} \left(2 \frac{h^2(1-f)^2}{r^2} + h'^2 \right), \quad (33)$$

$$\partial_i (\Phi^\dagger \Phi) \partial_i (\Phi^\dagger \Phi) = \nu^4 h^2 h'^2, \quad (34)$$

$$\Phi^\dagger D_i \Phi (D_i \Phi)^\dagger \Phi = \frac{\nu^4}{4} \left(\frac{h^4(1-f)^2}{r^4} (x^2 + y^2) + h^2 h'^2 \right). \quad (35)$$

Because $f \propto r^2$ and $h \propto r$ at small r , all of these terms are nonsingular at $r = 0$. The last term, and only the last term, is not spherically symmetric. The departure from spherical symmetry arises because the Dirac equation in the presence of the Sphaleron is not spherically symmetric when only one flavor of quark has a mass [11]. DPSSG evaded this problem by giving both flavors equal masses, which restores the symmetry. As we have seen, giving both quark flavors masses introduces a new dimension 6 operator, whose free energy density in the Sphaleron background is

$$\Phi^\dagger D_i i \tau_2 \Phi^* (D_i i \tau_2 \Phi^*)^\dagger \Phi = \frac{\nu^4 h^4(1-f)^2}{4 r^4} (r^2 + z^2). \quad (36)$$

The coefficients of this term and (35) are such that, when $g_t^2 = g_b^2$, the z^2 combines with the $x^2 + y^2$ to restore the spherical symmetry of the terms.

The approximation of DPSSG differs from the results with only the top quark massive by the contribution of Eq.(36) and is accurate only when this term is small. This is the case only when derivative dimension 6 operators give very small contributions, which is precisely the case where dimensional reduction is accurate.

To compute the Sphaleron energy it is convenient to follow [5] and introduce a dimensionless radial distance $\xi = g_w \nu r$. The contribution to the Sphaleron energy from the effective potential is then

$$\frac{4\pi\nu}{g_w} \int_0^\infty d\xi \xi^2 \frac{V(h\nu)}{g_w^2 \nu^4}. \quad (37)$$

The contribution from kinetic energy terms, including the fermions' contribution, is

$$\frac{4\pi\nu}{g_w} \int_0^\infty d\xi \left(1 + \frac{N_d g_w^2 D 4}{3} \right) \left(4f'^2 + 8 \frac{f^2(1-f)^2}{\xi^2} \right) + \left(1 + N_c g_t^2 D 4 \right) \left(h^2(1-f)^2 + \frac{\xi^2 h'^2}{2} \right) \quad (38)$$

where derivatives are with respect to ξ .

The contribution from Eq. (18) is

$$\frac{4\pi\nu}{g_w} \frac{7\zeta(3)g_w^4\nu^2}{128\pi^4 T^2} \int_0^\infty d\xi \xi^2 \left(-\frac{N_d}{180} \left[96 \frac{f'^2 f(1-f)}{\xi^4} \right] - \frac{N_d}{15} \left[8 \frac{f''^2}{\xi^2} - 32 \frac{f'' f(1-f)}{\xi^4} + 32 \frac{f^2(1-f)^2}{\xi^6} \right] \right). \quad (39)$$

We can get the other dimension 6 operators from Eqs. (14 - 18) and Eqs. (29 - 35) by always replacing $D6$ with $7\zeta(3)g_w^4\nu^2/128\pi^4 T^2$, removing g_w 's, replacing g_t with g_t/g_w and r with ξ , and integrating over $(4\pi\nu/g_w) \int \xi^2 d\xi$. For $\nu \sim T$, $D6$ is very small and dimension 6 operators have the parametric appearance of two loop effects.

From the discussion after the introduction in [4] we see that we want to find the configuration of form (21) with minimum free energy, that is the configuration which maximizes $(\Pi_{\text{zero mode cont.}})\kappa \exp(-E_{Sph}/T)$. This is a formidable task, since we cannot compute all of these terms analytically; in particular the zero mode contributions and part of the zero Matsubara frequency bosonic contribution to κ are beyond our analytic abilities. Fortunately, the Sphaleron is a saddlepoint configuration, and we should get almost exactly the right free energy if we include the dominant effects in the computation of the field configuration $f(\xi)$ and $h(\xi)$, and then compute the other corrections holding f and h fixed. This is because, as a saddlepoint, the free energy of the Sphaleron only changes quadratically with small changes to h and f .

To illustrate this point consider the Sphaleron configuration computed from a tree level Lagrangian $F_{ij}^a F_{ij}^a/4 + \lambda(\Phi^\dagger \Phi - \nu^2/2)^2$ and suppose that we are only interested in the correction fermions induce in the gauge fields, $N_d g_w^2 D4 F_{ij}^a F_{ij}^a/12$. We will choose $\mu = m_W(T = 0)$ in the proper time scheme and $T = 100$ GeV, so $D4 \simeq -0.0135$. Solving for the Sphaleron using the tree action we find $E_{Sph} = 35.0975\nu$, and estimating the fermionic correction as

$$\frac{4\pi\nu}{g_w} \int_0^\infty d\xi \left(\frac{N_d g_w^2 D4}{3} \right) \left(4f'^2 + 8 \frac{f^2(1-f)^2}{\xi^2} \right) \simeq -0.4433\nu \quad (40)$$

we get a total free energy of 34.6542ν . When we solve for the Sphaleron configuration, including the fermionic contribution as well as the tree terms, we find the Sphaleron energy, which now includes the fermionic correction, is 34.6520ν . The correction from this fermionic contribution is about 1% of E_{Sph} , and the error in estimating it at fixed f and h is about 0.01%, which is quadratic, as expected.

Obviously, though, this will not do when a correction is actually substantial and the modification of the free energy is comparable to E_{Sph} . This is potentially the case for the fermionic correction to the $(\Phi^\dagger \Phi)^2$ term in the effective potential. For the above parameters, the coefficient of the correction is $N_c g_t^4 D4 = -0.040$, which is to be compared with $\lambda(\mu) = 0.050$. Because of an unfortunate choice for μ , the “correction” for top quarks is almost as large as the tree level term itself, and we certainly cannot trust a result in which f and h are computed with the tree term and used to compute the quark fluctuation determinant. The only consistent, renormalization point independent thing to do is to include those fermionic contributions which correct operators appearing in the tree level action in the calculation of the Sphaleron configuration, that is to use the μ independent quantity $\lambda(\mu) + N_c g_t^4 D4$ in the calculation of f , h , and E_{Sph} . This is particularly important for the scalar self-coupling, because as mentioned earlier it is unprotected from large radiative corrections and in fact the top quark contribution here is substantial.

To work in terms of physical quantities, we first find a relation between the couplings at the scale μ and vacuum, physical masses. That is, we find $\lambda(\mu)$ in terms of $m_H(T = 0)$. We perform this calculation, including all one loop, fermionic contributions, in Appendix A. We ignore the bosonic corrections because we are most interested in understanding the fermionic radiative corrections in this paper, and because the bosonic corrections are smaller by a factor of $9m_W^4/12m_t^4 \simeq 1/30$. From the Appendix A results we find $\lambda(\mu) = (m_H^2/2\nu_o^2) - (N_c g_t^4/16\pi^2)(\ln(\mu^2/m_t^2) - a)$, where $a = 0$ in $\overline{\text{MS}}$ and $a = \gamma_E$ in proper time regulation. The quantity we should use in calculating the Sphaleron configuration at a

temperature T is

$$\lambda_T \equiv \lambda(\mu) + N_c g_t^4 D4, \quad (41)$$

which has no μ or renormalization scheme dependence, as we can note by inspecting Eq. (5) or Eq. (20)³.

A particularly convenient choice for μ is

$$\mu = \exp(-\gamma_E) \pi T, \text{ } \overline{\text{MS}} \text{ or } \mu = \exp(-\gamma_E/2) \pi T, \text{ proper time} \quad (42)$$

because in this case $\lambda_T = \lambda(\mu)$ and the Higgs field and gauge field wave functions take their tree values; but there is no need to choose this scale, as long as we compute the Sphaleron configuration and energy using couplings and wave functions corrected by the fermion contributions as in equations (38,41). This is precisely the prescription of the dimensional reduction program. We can test its reliability by seeing how large the remaining corrections, those coming from dimension 6 operators, are.

Of course, it is also necessary to include corrections from zero Matsubara frequency bosons, which are expected to be considerable and cannot be evaluated with a derivative expansion, as we have already discussed. The largest part of this correction comes from the cubic effective potential terms. The residual correction when this is completely removed is $\ln \kappa \simeq 1.5$ [7, 16] which is small enough that we can trust the computation in terms of fixed f and h as discussed above; but the effective potential contribution is large (at $T \simeq T_c$ it changes the very nature of the phase transition and should be considered an order 1 correction to E_{Sph}) and should be included in the evaluation of the Sphaleron configuration, as advocated in [17]. But before we can compute the Sphaleron energy including these corrections we must discuss the nature of the phase transition to find out at what temperature the baryon erasure begins.

4 Phase transition, Sphaleron energy

Let us briefly review the electroweak phase transition. At one loop the zero Matsubara frequency bosonic excitations generate negative cubic terms in the effective potential, which becomes ($\Theta_W = 0$) [18]

$$V(\nu) = -\frac{g_* \pi^2 T^4}{90} + \left(-\frac{m_0^2}{2} + \frac{(4g_t^2 + 3g_w^2 + 8\lambda)T^2}{32} \right) \nu^2 - \frac{g^3 T}{16\pi} \nu^3 - \frac{g_w^3 T}{4\pi} \left(\frac{11T^2}{6} + \frac{\nu^2}{4} \right)^{\frac{3}{2}} - \frac{3m_2^3 + m_1^3}{12\pi} + \lambda_T \frac{\nu^4}{4} + \text{dimension six}, \quad (43)$$

where $m_1^2 = \lambda_T \nu^2 / 2 + V''(\nu = 0)$, $m_1^2 = \lambda_T \nu^2 + m_2^2$, $g_* = 106.75$ is the number of radiative degrees of freedom, and “dimension six” means the higher terms found in Eq. (4). At high

³The alert reader may notice that $\lambda(\mu)$ also contains a term $(m_H^2/2\nu_o^2)(N_c g_t^2/16\pi^2)(\ln(\mu^2/m_t^2) - a)$, which apparently spoils the cancellation of the μ dependence discussed here. This is true, and it gets another correction because ν_o is μ dependent; but the mass squared parameter m_0^2 of the effective potential contains a similar μ dependence, so the effect of the correction is just to shift the location of the minimum of V by a proportional amount, without changing its height. The Higgs field wave function has the same dependence, so this has no influence on the Sphaleron energy.

temperature V has only one minimum at $\nu = 0$, but as temperature drops the negative cubic terms generate a second “asymmetric” minimum. At some temperature the second minimum becomes more thermodynamically favorable, and bubbles of the asymmetric phase begin to nucleate and grow shortly thereafter. The temperature at which the nucleations become common can be computed by standard techniques [19]. The expanding bubbles liberate latent heat, so that the temperature after the transition is somewhat higher than the nucleation temperature. The temperature the plasma reheats to due to this latent heat is determined by the condition that the broken phase energy density $E = V - T\partial V/\partial T$ must equal the symmetric phase energy density at the temperature where the nucleations occurred. It is at this reheat temperature, immediately after the phase transition, that quasi-equilibrium erasure of any baryon number excess generated at the phase transition begins. It continues for all times thereafter, but as Eq. (22) shows, the rate depends strongly on the Sphaleron energy, which changes rapidly with T , as ν moves towards its zero temperature value. Almost all the baryon erasure takes place within a fraction of a Hubble time, so the Sphaleron rate is only relevant at temperatures quite close to the reheat temperature.

We have computed the reheat temperature and the Sphaleron energy at the reheat temperature for a number of physical Higgs masses, using the 1 loop relations between Higgs mass and λ presented in Appendix A and including the negative cubic effective potential terms from zero Matsubara frequency bosonic modes. (We always use $g_w = 0.65$ and take the Weinberg angle to be zero. In fact, to account for the quark contribution to the gauge field wave function, the value of g_w we should use is $g_w(\mu) - N_d g_w^3 D 4/6$, which is T dependent. The T dependence is very mild, behaving as $g_w * (N_d g_w^2 / 48\pi^2 \simeq 0.01) \ln T$. Since this dependence is so weak, and since we are not including the influence of bosons’ nonzero Matsubara modes, which will contribute an opposite and slightly larger temperature dependence [9], we will not worry about it. However we will take full consideration of the correction to λ , which as discussed is not at all weak.) The results, together with the contributions from dimension 6 operators, are presented in table 1. The ν^6 contribution to the effective potential was computed by inclusion in the calculation of the Sphaleron configuration, and the others were performed perturbatively. It is clear from the table that, as expected, dimension 6 operators make only a tiny contribution to the Sphaleron energy. Hence we can conclude that near the phase transition temperature the expansion is very well behaved and the use of the dimensionally reduced theory is well justified. The largest dimension 6 correction comes from the effective potential term, owing to the high power of g_t ; this and only this term may not be completely negligible, increasing the Sphaleron energy and the strength of the phase transition by a few percent for light Higgs. Generally the terms become progressively less important as they contain more derivatives. In particular the very small contribution from $(D^2\Phi)^\dagger D^2\Phi$ gives confidence that the expansion of Eq. (8) in derivatives is justified.

We have not continued the table down below 30 GeV partly because this range is experimentally excluded, partly because questions of vacuum stability become increasingly hard to avoid in this range, and partly because the results barely differ from those at $m_H = 30$ GeV. This is because the one loop relations for the Higgs mass give nonzero λ_T even as the physical Higgs mass $m_H \rightarrow 0$.

The next step is to use these results to determine the baryon number depletion. From

Eq.(22) we find that baryon number is depleted by a factor of

$$\exp \left(\int_{t_{\text{reheat}}} 13N_F T \left(\frac{\alpha_w}{4\pi} \right)^4 \frac{\omega_-}{2m_W} \mathcal{N}_{tr} \mathcal{N} \mathcal{V}_{rot} \left(\frac{4\pi\nu}{g_w T} \right)^7 \kappa \exp \frac{-E_{Sph}}{T} dt \right) \quad (44)$$

The elapsed time is related to the change in temperature by $dt = dT/(HT)$, with H the Hubble constant, which is approximately $H = T(8\pi^3 g_*/90)^{1/2}(T/m_{pl})$. To perform the integral we must numerically repeat the evaluation of the Sphaleron energy at many values of T close to the reheat temperature. Most of the coefficients in Eq. (44) are given in [15]. We use the values found there for ω_- and the \mathcal{N} , even though they were computed for slightly different f and h , because the product of these factors turns out to be very insensitive to changes in the configuration [15]. The other factor we require is the difference between the Sphaleron energy which we have calculated and the (1 loop) value of $E_{Sph} - T \ln \kappa$. This is due to derivative corrections from zero Matsubara frequencies and can be approximated from the results of Baacke and Junker [7], who find that, when the full tadpole is removed (ie. all effective potential contributions are subtracted out), $\ln \kappa$ is about 1.5, independent of ν and quite weakly dependent on λ_T . (Again, this quantity was computed for different f, h in that paper, but again it proved to be quite insensitive to configuration, so we will use this value. This introduces an uncertainty in our final value for the erasure rate of perhaps ± 1 in the exponential.) We can then perform the integral in Eq. (44) numerically. We find that almost all of the erasure occurs in a range of T less than 0.5% from the reheat temperature, essentially because ν changes very rapidly with temperature immediately after the transition. This narrows the available time for the Sphaleron erasure, but even for $m_H = 30$ GeV we find that the baryon number is depleted by a factor of about $\exp(11.3)$. If the conjecture about the effect of Landau damping on the negative frequency mode in [4] is correct then the suppression is smaller by about 2 in the exponent. The results for several Higgs masses are presented in the table; in all cases the erasure is very considerable. Since the most optimistic estimates of baryogenesis in the minimal standard model can barely account for the current abundance [20], this apparently rules out electroweak physics as the source of baryogenesis in the minimal standard model.

5 Conclusion

It appears that a perturbative treatment of fermions is very well justified near the phase transition temperature, and that dimensional reduction should be accurate, though the correction from the dimension 6 contribution to the effective potential may not be completely negligible.

How should we understand the results of DPSSG in light of this conclusion?

First recall what we have done here. We find that at a general μ the fermions will induce nonzero corrections to couplings and wave functions, and in particular the correction to λ is potentially large. Following the idea of the dimensional reduction program we combine these contributions with the tree couplings, resulting in renormalization point independent couplings λ_T , which are used to compute the Sphaleron energy. The Sphaleron energy then already contains that part of the fermion fluctuation determinant which is

understood as coupling and wave function corrections; the residual correction, which comes from dimension 6 (and higher) operators, is explicitly found to be very small. However, had we used the tree couplings at some scale μ , we would then expect fermions to give a (μ dependent) nonzero correction due to the difference $\lambda_T - \lambda(\mu)$. Only at one particular, convenient renormalization point (Eq. (42)) would this contribution vanish.

Diakonov et. al. use a fixed μ (in [11] they use $\mu = m_W$ in the proper time scheme, and in [21] they use $\mu \simeq m_t \exp(\gamma_E/2)$) and compute the Sphaleron configuration from the tree Lagrangian (corrected however by the thermal contributions to the $\Phi^\dagger \Phi$ term). We then expect from our work that they should find a correction arising from the difference $\lambda_T - \lambda(\mu)$ equal to

$$\frac{4\pi\nu}{g_w} \frac{\lambda_T - \lambda(\mu)}{g_w^2} \int_0^\infty d\xi \xi^2 \frac{h^4 - 1}{4}, \quad (45)$$

which will depend on the top mass as g_t^4 , exactly as they find. This is not a contradiction of the dimensional reduction scheme, which predicts that, because of their use of $\lambda(\mu)$ with their choice of μ , they should find such a term. We should note that, when this term is large (as it is for the choice of μ made in [11]), one should not trust the functions h, f computed from the tree Lagrangian but should include this radiative correction in their computation, as we have done here. In other words, the numerical work of Diakonov et. al. is probably accurate, but because of the way they have done the problem they will not necessarily produce the free energy of the configuration which actually limits the baryon erasure rate.

We should also note that in [11] Diakonov et. al. use a tree, rather than 1 loop, relation between m_H and $\lambda(\mu)$, and a tree, rather than 1 loop, value of λ in calculating the phase transition temperature. (In [21] the relation between m_H and $\lambda(\mu)$ is computed at one loop, but the transition temperature is still found using $\lambda(\mu)$ rather than λ_T .) This is inconsistent with a one loop analysis of the Sphaleron rate and may explain the difference in our results for the dissipation of baryon number.

What is the overall effect of fermions? For small Higgs mass, $\lambda_T > \lambda_{\text{tree}}$, and as $\nu(T_{\text{reheat}})$, and hence E_{Sph} , fall with increasing λ_T , we find more baryon number dissipation than we would if we ignored fermions altogether and used λ_{tree} . This is the reason that, even for very small Higgs mass, we still find substantial baryon number erasure. We should emphasize once more that to find this result it was important to apply one loop corrections systematically, in the effective potential, the phase transition temperature, and the Sphaleron energy, but that the effect is basically perturbative, and the high temperature expansion accounts for it successfully.

Do the results of the last section preclude electroweak baryogenesis? They make it unlikely that baryogenesis can be viable in the minimal standard model. However we should note that we have only used the 1 loop effective potential, and while extending our results to the two loop potential, which is known, gives essentially the same conclusions, it is not clear that the perturbative treatment of the effective potential is reliable; the phase transition may be stronger than perturbation theory suggests. Also, we have said nothing about extensions to the standard model. For instance, in the two doublet model, the phase transition can be much stronger without contradicting experimental bounds on the Higgs mass [22], and the Sphaleron bound only narrows the parameter space.

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6 Appendix A: Proper Time Calculations

In this appendix we will show how to compute the finite temperature fluctuation determinant using a proper time regulation analogous to that of DPSSG. The basic idea is to use the relationship

$$\ln \frac{\text{Det}K}{\text{Det}K_0} = \lim_{\epsilon \rightarrow 0} -\text{Tr} \int_{\epsilon}^{\infty} \frac{dt}{t} (e^{-Kt} - e^{-K_0 t}), \quad (46)$$

which holds when K is a quadratic, positive definite operator and K_0 is its free, massless approximation. In our case the operator $\gamma^\mu \partial_\mu - m$, or Eq. (1), is not quadratic or positive definite. Fortunately it has the same spectrum as the operator $-\gamma^\mu \partial_\mu - m$, so we can write the determinant we actually want in terms of a quadratic, positive definite operator,

$$\begin{aligned} -\ln \text{Det}(\gamma^\mu \partial_\mu - m) &= -\frac{1}{2} (\ln \text{Det}(\gamma^\mu \partial_\mu - m) + \ln \text{Det}(-\gamma^\mu \partial_\mu - m)) \\ &= -\frac{1}{2} \text{Tr} \ln(-\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - \gamma^\mu (\partial_\mu m) + m^2), \end{aligned} \quad (47)$$

which acts on a Euclidean space where the time direction is cyclic with antiperiodic boundary conditions and period T . It therefore already includes all thermal effects, which do not have to be added by hand, as in the treatment of DPSSG. However, we will have to be careful when we regulate to see that we are making T independent subtractions.

Following [23] we perform the trace by inserting a complete set of plane-wave states,

$$\text{Tr} e^{-Kt} = \text{tr} \int d^4x T \sum_{p_0} \int \frac{d^3p}{(2\pi)^3} e^{-ip \cdot x} e^{-Kt} e^{ip \cdot x}, \quad (48)$$

where tr is over Dirac indices. The factor $\exp(ip \cdot x)$ can be brought through the operator to cancel $\exp(-ip \cdot x)$, but in doing this all derivative operators in K are shifted, $\partial_\mu \rightarrow \partial_\mu + ip_\mu$. The zero temperature limit is found by replacing the sum on p_0 with the integral $\int dp_0/(2\pi)$.

We illustrate the renormalization procedure by computing the effective potential in this regulation. In this case $K = -\partial^2 + m^2$ because the mass is space independent. After the shift, the derivatives have nothing on which to act and do not contribute; we can drop them. The problem is then to compute

$$2 \int_{\epsilon}^{\infty} \frac{dt}{t} \int d^3x \sum_{p_0} \int \frac{d^3p}{(2\pi)^3} e^{-p^2 t} e^{-p_0^2 t} (e^{-m^2 t} - 1), \quad (49)$$

where we have performed the Dirac trace and removed the trivial integral over dx_0 so that the result will be the free energy density.

Next we expand $\exp(-m^2 t)$ in powers of t . The first term is cancelled by the -1 . The second gives

$$-2 \int m^2 d^3 x \int_{\epsilon}^{\infty} dt T \sum_{p_0} e^{-p_0^2 t} \int \frac{d^3 p}{(2\pi)^3} e^{-p^2 t}. \quad (50)$$

This expression is small t divergent, cut off by ϵ . To render the theory cutoff independent we should add and subtract a temperature independent expression with the same small t behavior and absorb the one we added with a counterterm in the tree level mass. The correct expression to subtract is

$$-2 \int m^2 d^3 x \int_{\epsilon}^{\infty} dt \int \frac{dp_0}{2\pi} e^{-p_0^2 t} \int \frac{d^3 p}{(2\pi)^3} e^{-p^2 t}. \quad (51)$$

There is now no obstacle to performing the integral over t , or to setting ϵ equal to zero. The result is

$$-2 \int m^2 d^3 x \int \frac{d^3 p}{(2\pi)^3} \left(T \sum_{p_0} - \int \frac{dp_0}{2\pi} \right) \frac{1}{p^2 + p_0^2}. \quad (52)$$

Both the sum and the integral over p_0 are straightforward, giving

$$\sum_{p_0} \frac{1}{p_0^2 + p^2} = \frac{\tanh \frac{p}{2T}}{2p}, \quad \int \frac{dp_0}{2\pi} \frac{1}{p_0^2 + p^2} = \frac{1}{2p}',; \quad (53)$$

combining them and performing the integral over angles, we get

$$-2 \int d^3 x m^2 \frac{1}{2\pi^2} \int \frac{-1}{\exp(p/T) + 1} p dp = \int d^3 x \frac{m^2 T^2}{12}, \quad (54)$$

which is the well known expression for the thermal contribution to the Higgs mass squared.

In their discussion of the renormalization of the theory DPSSG advocate cutting off the proper time integral of the counterterm at some finite upper bound, call it μ^{-2} (in their case, m_W^{-2}). Doing so changes the result of the above calculation to

$$\frac{m^2 T^2}{12} - \frac{\mu^2 m^2}{8\pi^2}, \quad (55)$$

which means there is a discrepancy between the tree level mass squared parameter and the renormalized, vacuum mass squared parameter. There is nothing wrong with this in principle as long as we remember it is there; we should use the renormalized mass squared when performing calculations such as the Sphaleron configuration and remember that we have already included part of the fermion contribution by doing so; we will need to subtract it off, along with the thermal mass squared. It is much easier and more straightforward, however, to follow our procedure and absorb the entire vacuum correction in a counterterm.

Continuing to expand $\exp(-m^2 t)$, the next term is

$$+ \int m^4 d^3 x \int_{\epsilon}^{\infty} t dt \sum_{p_0} e^{-p_0^2 t} \int \frac{d^3 p}{(2\pi)^3} e^{-p^2 t}, \quad (56)$$

which is logarithmically divergent at small t . Because the lowest Matsubara frequency is nonzero, it is cut off exponentially at large t and there are no infrared problems in

its evaluation. Again we have an available Lagrangian parameter in which to absorb the divergence, and we should add and subtract a T independent expression with the same ultraviolet behavior, and absorb the one with the same sign into the Higgs self-coupling parameter. The corresponding vacuum integral is both ultraviolet and infrared divergent, and to prevent infrared divergences we must introduce a renormalization scale into the problem. Following DPSSG, we subtract

$$\int m^4 d^3x \int_{\epsilon}^{\mu^{-2}} t dt \int \frac{dp_0}{2\pi} e^{-p_0^2 t} \int \frac{d^3p}{(2\pi)^3} e^{-p^2 t}, \quad (57)$$

giving

$$\int m^4 d^3x \left(\int_{\epsilon}^{\infty} dt T \sum_{p_0} - \int_{\epsilon}^{\mu^{-2}} dt \int \frac{dp_0}{2\pi} \right) t e^{-p_0^2 t} \int \frac{d^3p}{(2\pi)^3} e^{-p^2 t}. \quad (58)$$

The integral over d^3p gives $t^{-3/2}/(8\pi^{3/2})$. The counterterm can then be evaluated directly and gives $+\ln(\mu^2\epsilon)/(16\pi^2)$. The remaining integral,

$$\frac{T}{8\pi^{\frac{3}{2}}} \sum_{p_0} \int_{\epsilon}^{\infty} t^{-\frac{1}{2}} dt e^{-p_0^2 t}, \quad (59)$$

is more delicate. For small terms in the sum the integral over t is approximately $\sqrt{\pi}/p_0$, so the early terms in the sum are

$$\frac{T}{4\pi} \sum_{l=1,3,\dots} \frac{1}{l\pi T}. \quad (60)$$

The difference between this sum and a corresponding integral is concentrated in the first few terms. The sum is approximately

$$\frac{1}{4\pi} \left[\left(\int_{\pi T} \frac{dp_0}{2\pi} \frac{1}{p_0} \right) + \frac{\gamma_E + \ln 2}{2\pi} \right], \quad (61)$$

so smoothing over the sum in Eq. (59) introduces a correction of $(\gamma_E + \ln 2)/(8\pi^2)$, giving

$$\frac{\gamma_E + \ln 2}{8\pi^2} + \frac{1}{4\pi^{\frac{3}{2}}} \int_{\pi T}^{\infty} \frac{dp_0}{2\pi p_0} \int_{\epsilon}^{\infty} \frac{dt}{t^{\frac{1}{2}}} e^{-p_0^2 t}. \quad (62)$$

The integral over p_0 can be performed by parts, giving

$$\frac{\gamma_E + \ln 2}{8\pi^2} + \frac{1}{4\pi^{\frac{3}{2}}} \frac{-1}{2\sqrt{\pi}} \left(\ln \pi T + \frac{1}{2} \ln \epsilon - \frac{1}{2} \psi(1/2) \right), \quad (63)$$

with ψ the digamma function, $\psi(1/2) = -\gamma_E - 2\ln 2$. The result is then

$$\int m^4 d^3x \frac{\gamma_E - 2\ln \pi + \ln(\mu^2/T^2)}{16\pi^2}. \quad (64)$$

The fraction in the integral gives the proper time renormalization value of $D4$.

The terms higher order in m^2 are actually easier; the term at order m^{2n} is given by the integral

$$2 \int_0^{\infty} \frac{dt}{t} \int d^3x \frac{(-1)^n}{n!} t^n m^{2n} \sum_{p_0} \int \frac{d^3p}{(2\pi)^3} e^{-p^2 t} e^{-p_0^2 t}. \quad (65)$$

Performing the integral over t , we get

$$\frac{2(-1)^n}{n} \int d^3x m^{2n} \sum_{p_0} \int_0^\infty \frac{p^2 dp}{2\pi^2} \frac{1}{(p^2 + p_0^2)^n}. \quad (66)$$

The result per unit volume is

$$\frac{(-1)^n m^{2n}}{n\pi^2} \left(\sum_{p_0} p_0^{3-2n} \right) \int_0^\infty \frac{y^2 dy}{(y^2 + 1)^n}, \quad (67)$$

from which Eq. (4) follows immediately by performing the sum and the integral. Note that the calculation was completely large t finite because the $\exp(-p_0^2 t)$ term always decays exponentially fast, since p_0^2 is always at least $(\pi T)^2$. This is to be contrasted with the zero temperature [23] and zero Matsubara frequency bosonic [15] cases and explains why the expansion in operator dimension is possible at finite temperatures for fermions, while it is known to have trouble at zero temperature and for the zero Matsubara frequency bosonic modes.

The same basic techniques can be used for the case with gauge fields and spatial variation. A complication which arises when computing other dimension 4 operators is that, in contributions involving A_0 , powers of p_0 appear in addition to $\exp(-p_0^2 t)$. The result is that the argument of the digamma function shifts by 1 per power of p_0^2 . This is why the dimension 4 terms containing A_0 are not simply multiples of $D4$.

Next we will compute the fermionic contribution to the vacuum effective potential in this regulation. From the above, the contribution from each color is

$$2 \left[\left(\int_\epsilon^\infty \frac{dt}{t} \int \frac{d^4 p}{(2\pi)^4} e^{-p^2 t} (e^{-m^2 t} - 1 + m^2 t) \right) - \left(\int_\epsilon^{\mu^{-2}} \frac{dt}{t} \int \frac{d^4 p}{(2\pi)^4} e^{-p^2 t} \frac{m^4}{2} t^2 \right) \right]. \quad (68)$$

The -1 is the vacuum energy subtraction, the $m^2 t$ is the mass squared counterterm, and the last expression is the self-coupling counterterm. The integrals over p may be performed immediately, giving

$$\frac{1}{8\pi^2} \left(\int_\epsilon^\infty \frac{dt}{t^3} (e^{-m^2 t} - 1 + m^2 t) \right) - \frac{1}{8\pi^2} \frac{m^4}{2} \int_\epsilon^{\mu^{-2}} \frac{dt}{t}. \quad (69)$$

After integrating the first expression by parts three times, we get

$$\frac{m^4}{16\pi^2} \left(\frac{3}{2} + \ln \frac{\mu^2}{m^2} - \gamma_E \right). \quad (70)$$

Recalling that $m^2 = g_t^2 \Phi^\dagger \Phi$, summing on colors, and adding this expression to the tree level effective potential

$$-m_0^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \quad (71)$$

we find that the curvature of the effective potential (the second derivative with respect to ν) at the minimum ($\nu = \nu_o$) is

$$V'' = 2\nu_o^2 \left(\lambda + \frac{N_c g_t^4}{16\pi^2} \left(\ln \frac{\mu^2}{m_t^2} - \gamma_E \right) \right), \quad (72)$$

and the effective potential parameter m_0^2 is

$$m_0^2 = \frac{V''}{2} + \frac{N_c g_t^4 \nu^2}{16\pi^2}. \quad (73)$$

The analogous $\overline{\text{MS}}$ result is the same but with the $-\gamma_E$ removed from Eq. (72).

V'' is not the physical Higgs mass squared. The physical Higgs mass is the ratio of potential to kinetic energy of a long wavelength fluctuation about the minimum, times the wave number of the fluctuation; there is a wave function correction which requires computing the self-energy at the pole mass. We have been unable to do the calculation in proper time regulation (see however [21]), but we have performed it in $\overline{\text{MS}}$ using an integral from [24]. The result is that

$$m_H^2 = V'' \left(1 - \frac{N_c g_t^2}{16\pi^2} \left(\ln \frac{\mu^2}{m_t^2} + 2 - \frac{2\sqrt{4m_t^2 - m_H^2}}{m_H} \arctan \frac{m_H}{\sqrt{4m_t^2 - m_H^2}} \right) \right). \quad (74)$$

The proper time value is presumably the same but with a $-\gamma_E$ inserted after the log. For $4m_t^2 \gg m_H^2$ the expression following the log is about $2m_H^2/3(4m_t^2 - m_H^2) \simeq 0$ and we have permitted ourselves to neglect it when relating the quartic coupling to the physical Higgs mass.

7 Appendix B: Other high dimension operators

We list here those dimension 4 and 6 operators induced by fermions which contain both weak $SU(2)$ and strong $SU(3)$ fields. These are probably of no phenomenological consequence and almost certainly do not influence the strength of the phase transition. We include them only for completeness.

We write the time component of the gluon field as A_{0g} and the field strength tensor as G_{ij} . Denoting the number of families as $N_F = 3$, the dimension four, mixed terms are

$$\frac{-g_t^2 g_s^2}{8\pi^2} A_{0g}^2 \Phi^\dagger \Phi - \frac{g_w^2 g_s^2 N_F}{16\pi^2} A_0^2 A_{0g}^2. \quad (75)$$

The dimension six mixed terms are

$$\begin{aligned} & 2g_t^2 g_s^2 D6 A_{0g}^2 (D_i \Phi)^\dagger D_i \Phi + g_t^4 g_s^2 D6 A_{0g}^2 (\Phi^\dagger \Phi)^2 + \frac{g_s^2 g_w^2 g_t^2 D6}{8} A_{0g}^2 A_0^2 \Phi^\dagger \Phi \\ & + \frac{g_s^2 g_w^2 N_F D6}{3} A_{0g}^2 (D_i A_0)^a (D_i A_0)^a + \frac{g_s^2 g_w^2 N_F D6}{6} A_{0g}^2 F_{ij}^a F_{ij}^a \\ & + \frac{g_s^2 g_w^2 N_F D6}{3} (D_i A_{0g})^a (D_i A_{0g})^a A_0^2 + \frac{2g_s^2 g_t^2 D6}{3} \partial_i (A_{0g}^2) \partial_i (\Phi^\dagger \Phi) \\ & + \frac{g_s^2 g_w^2 N_F D6}{3} \partial_i (A_{0g}^2) \partial_i (A_0^2) - \frac{g_s^2 g_t^2 D6}{3} G_{ij}^a G_{ij}^a \Phi^\dagger \Phi + \frac{g_w^2 g_s^2 N_F D6}{6} G_{ij}^a G_{ij}^a A_0^2. \end{aligned} \quad (76)$$

There are also pure glue terms which are identical to the pure gauge terms listed in the text, but with the substitution $N_d \rightarrow 2N_f$ where $N_f = 6$ is the number of quark flavors. The 2 is because both right and left handed quarks couple to the gluons, whereas the weak coupling is chiral.

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m_H (GeV)	30	50	60
T_{reheat} (GeV)	71.2127	86.6495	95.6517
E_{Sph}/T	29.18	24.79	22.38
E_{Sph}/T from	$(\Phi^\dagger \Phi)^3$	1.6	0.60
	$\Phi^\dagger \Phi (D_i \Phi)^\dagger D_i \Phi$	-6.3×10^{-3}	-3.7×10^{-3}
	$\partial_i (\Phi^\dagger \Phi) \partial_i (\Phi^\dagger \Phi)$	-3.3×10^{-3}	-1.8×10^{-3}
	$(D_i \Phi)^\dagger \Phi \Phi^\dagger D_i \Phi$	-2.1×10^{-3}	-1.2×10^{-3}
	$F_{ij}^a F_{ij}^a \Phi^\dagger \Phi$	-9.9×10^{-4}	-6.2×10^{-4}
	$(D_i F_{ij})^a (D_k F_{kj})^a$	-4.6×10^{-4}	-2.9×10^{-4}
	$f_{abc} F_{ij}^a F_{jk}^b F_{ki}^c$	-8.4×10^{-5}	-5.3×10^{-5}
	$(D^2 \Phi)^\dagger D^2 \Phi$	-4.1×10^{-6}	-4.2×10^{-6}
	depletion factor	$\exp(11.3)$	$\exp(15.0)$
m_H (GeV)	70	80	90
T_{reheat} (GeV)	105.1149	114.8275	124.6283
E_{Sph}/T	20.49	19.26	18.70
E_{Sph}/T from	$(\Phi^\dagger \Phi)^3$	0.17	0.100
	$\Phi^\dagger \Phi (D_i \Phi)^\dagger D_i \Phi$	-2.0×10^{-3}	-1.6×10^{-3}
	$\partial_i (\Phi^\dagger \Phi) \partial_i (\Phi^\dagger \Phi)$	-8.8×10^{-4}	-6.7×10^{-4}
	$(D_i \Phi)^\dagger \Phi \Phi^\dagger D_i \Phi$	-6.3×10^{-4}	-4.9×10^{-4}
	$F_{ij}^a F_{ij}^a \Phi^\dagger \Phi$	-3.6×10^{-4}	-3.1×10^{-4}
	$(D_i F_{ij})^a (D_k F_{kj})^a$	-1.7×10^{-4}	-1.5×10^{-4}
	$f_{abc} F_{ij}^a F_{jk}^b F_{ki}^c$	-3.1×10^{-5}	-2.6×10^{-5}
	$(D^2 \Phi)^\dagger D^2 \Phi$	-4.8×10^{-6}	-5.5×10^{-6}
	depletion factor	$\exp(19.0)$	$\exp(20.3)$

Table 1: Reheat temperature, Sphaleron energy at the reheat temperature, and contribution of dimension 6 operators to Sphaleron energy for a number of physical Higgs masses

Figure 1: Feynman diagrams corresponding to two insertions of H_I .

